The Perron-Frobenius Theorem

$\mathcal{R}$ = $N \times N$ transition rate matrix $(N \geq 2)$

(connected graph; no directed edges)

$\mathcal{T} = e^{\mathcal{R}\Delta t}$ for some $\Delta t > 0$

$$\vec{1}^{+}\mathcal{R} = \vec{0}^{+} \implies \exists\, \vec{\pi} \text{ s.t } \mathcal{R}\vec{\pi} = \vec{0}$$

(We haven't yet shown that $\vec{\pi}$ is unique, or even that its elements must be real.)

$$\vec{1}^{+}\mathcal{T} = \vec{1}^{+} \quad , \quad \mathcal{T}\vec{\pi} = \vec{\pi}$$

The eigenvectors & eigenvalues of $\mathcal{R}$ are in 1-to-1 correspondence with those of $\mathcal{T}$:

$$\begin{array}{lcl} \mathcal{R}\vec{u} = \lambda\vec{u} & & \mathcal{T}\vec{u} = \mu\vec{u} \\ & \longleftrightarrow & \\ \vec{v}^{+}\mathcal{R} = \vec{v}^{+}\lambda & & \vec{v}^{+}\mathcal{T} = \vec{v}^{+}\mu \end{array}$$

$$\boxed{\mu = e^{\lambda \Delta t}}$$

The Perron-Frobenius Theorem (PFT) makes strong statements about the eigenvectors & eigenvalues of $\mathcal{T}$, which then translate to statements about $\mathcal{R}$.

Definitions

A matrix is non-negative (or positive) if all of its elements are non-neg. (pos.)

A non-negative square matrix M is primitive if $\exists\ k \geq 1$ s.t. $M^k > 0$.

(M^k is positive)

Thus every positive matrix is primitive, but not vice-versa.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

One of these is primitive, the other isn't.

Perron-Frobenius Thm. for primitive matrices.

If $M \geq 0$ is a primitive matrix then:

1. One of its eigenvalues is positive and greater than (in absolute value) all other eigenvalues:

$$\mu_{PF} \in \mathbb{R}, \quad \mu_{PF} > |\mu_i|$$

Perron-Frobenius eigenvalue aka <u>dominant</u> eigenvalue

all other eigenvalues

2. The corresponding left & right eig. vectors are positive:

$$M\vec{u} = \mu_{PF}\,\vec{u} \implies u_i > 0 \quad \forall i$$

etc. for left eig. vector $\vec{v}^{T}$

3. μ_{PF} is a <u>simple</u>, i.e. non-degenerate, root of the characteristic equation

$$|M - \mu I| = 0$$

Therefore the left & right eigenvectors corresponding to μ_{PF} are <u>unique</u>.

Also: if $M\vec{u}_i = \mu_i \vec{u}_i$, where $\mu_i \neq \mu_{PF}$, the $\vec{u}_i$ can't be positive, & similarly for $\vec{v}^\dagger M = \vec{v}^\dagger \mu_i$.

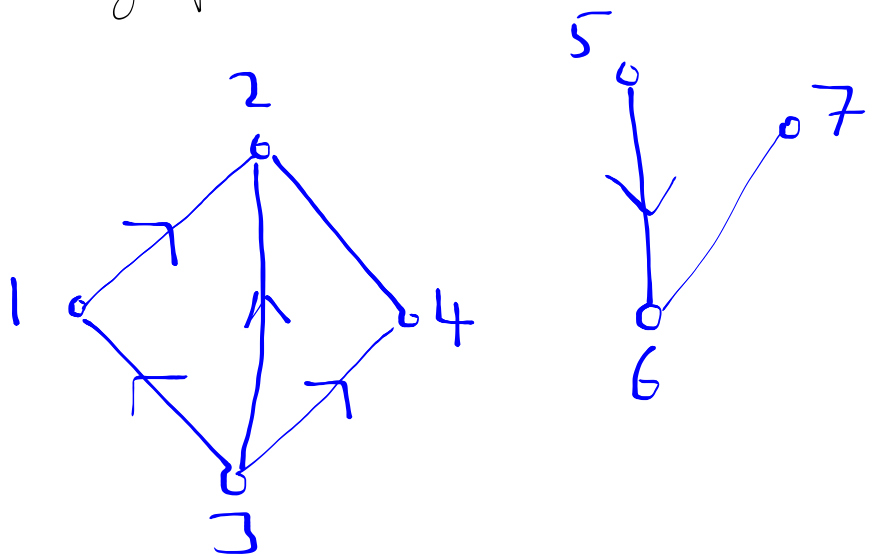
This follows because $\vec{v}_i^\dagger \cdot \vec{u}_{PF} = \vec{v}_{PF}^\dagger \cdot \vec{u}_i = 0$.

(any left eigenvector corresponding to one eigenvalue must be orthogonal to any right eigenvector corresponding to a different eig. val.)

Thus if we find a positive left or right eigenvector of M, then it must correspond to the P.F. eigenvalue.

It's instructive to work through an example that is not taken from physics ...

Consider the graph:



7 nodes, 7 edges (incl. directed edges)

not connected!

Imagine the following random walk:
(discrete time steps)

1. With probability α (where $0<\alpha<1$), move along an edge.

   eg. if starting from node 3, jump to 1, 2, or 4; if starting from 4, jump to 2.

2. With prob. $1-\alpha$, jump to 1 of the 7 states, selected randomly.

Discrete-time master equation:

$$\vec{p}(t+1) = M\,\vec{p}(t)$$

M_{ij} = prob. to jump to i, from j

$$= \alpha \cdot \frac{A_{ij}}{d_j} + (1-\alpha)\frac{1}{N} \qquad (A_{ii}=0)$$

A_{ij} = adjacency matrix

d_j = # of outgoing edges from j

$$= \sum_i A_{ij}$$

Since $M>0$, PF Thm. applies.

Easy to show: $\vec{1}^{\dagger} M = \vec{1}^{\dagger}$.

Thus, $\vec{1}^{\dagger}$ is a left eigenvector whose elements are all positive. We conclude that:

- $\mu_{PF} = 1$
- $\exists\ \vec{\pi} > 0$ s.t. $M\vec{\pi} = \vec{\pi}$

unique stationary prob. dist.

- $|\mu_i| < 1$, all other eig. vals.

$$\vec{p}(t) = M^t \vec{p}(0) \xrightarrow{t \to \infty} \vec{\pi}$$

This example illustrates the "random surfer model" for internet browsing.

$\pi_i \propto$ frequency with which site i is visited, in the long time limit ... provides a measure of the popularity / importance of i.

This model (or a very similar one) was used in the early Google PageRank algorithm.

Now return to the original problem

$$\frac{d\vec{p}}{dt} = R\vec{p} \; ,$$

assuming an <u>undirected</u>, <u>connected</u> network

$$\frac{d\vec{p}}{dt} = \mathbb{R}\vec{p} \quad , \quad \vec{1}^{+}\mathbb{R} = \vec{0}^{+} \quad , \quad \mathbb{R}\vec{\pi} = \vec{0} \quad , \quad \lambda_0 = 0$$

$$\mathbb{J} = e^{\mathbb{R}\Delta t} \quad , \quad \vec{1}^{+}\mathbb{J} = \vec{1}^{+} \quad , \quad \mathbb{J}\vec{\pi} = \vec{\pi} \quad , \quad \mu_0 = 1$$

> 0 ... PF Thm. applies $\qquad \mu_n = e^{\lambda_n \Delta t}$

$$\mu_{PF} = \mu_0 = 1 > |\mu_1| \geq |\mu_2| \geq \cdots$$

$$\mu_n = e^{\lambda_{nR}\Delta t} e^{i\lambda_{nI}\Delta t} \qquad \lambda_n = \lambda_{nR} + i\lambda_{nI}$$

$$|\mu_n| = e^{\lambda_{nR}\Delta t} < 1 \quad \text{for } n \geq 1$$

$$\therefore \boxed{\mathrm{Re}(\lambda_n) < 0}$$

recall same result for FP operator

So: $\vec{\pi}$ is the <u>unique</u> stationary state, but can we guarantee that

$$\lim_{t\to\infty} \vec{p}(t) = \vec{\pi} \ ?$$

<u>YES</u>.

Easy case: $\mathcal{R}, \mathcal{J}$ are diagonalizable.

0

$$\mathcal{R} = U \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} V^\dagger$$

$UV^\dagger = V^\dagger U = I$

$R = UDV^\dagger$

$\to R^k = UD^kV^\dagger$

1

$$e^{\mathcal{R}t} = U \begin{pmatrix} e^{\lambda_0 t} & & & \\ & e^{\lambda_1 t} & & \\ & & \ddots & \\ & & & e^{\lambda_N t} \end{pmatrix} V^\dagger$$

$$\xrightarrow{t\to\infty} U \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} V^\dagger$$

since $\text{Re}(\lambda_n) < 0$ for $n \geq 1$

$\vec{\pi}$

$$= \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ u_0 & u_1 & \cdots & u_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \begin{pmatrix} \longleftarrow & v_0^\dagger & \longrightarrow \\ \longleftarrow & v_1^\dagger & \longrightarrow \\ & \vdots & \end{pmatrix}$$

$\leftarrow \vec{1}^\dagger$

$$= \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{\pi} & \vec{\pi} & \cdots & \vec{\pi} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} = \vec{\pi}\, \vec{1}^\dagger \quad ; \quad e^{\mathcal{R}t} \vec{p} \to \vec{\pi} \left(\vec{1}^\dagger \cdot \vec{p}\right) = \vec{\pi}$$

$N \times 1$ $\quad$ $1 \times N$

Harder case: $\mathcal{R}, \mathcal{T}$ not diagonalizable

$$\mathcal{T} = e^{\mathcal{R}\Delta t}$$

$$= U \begin{pmatrix} 1 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_L \end{pmatrix} V^{+}$$

Jordan canonical form

$$J_\ell = \begin{pmatrix} \mu_\ell & 1 & & 0 \\ & \mu_\ell & 1 & \\ & & \ddots & 1 \\ 0 & & & \mu_\ell \end{pmatrix} \qquad |\mu_\ell| < 1$$

$$\lim_{n\to\infty} J_\ell^n = 0 \qquad \therefore \lim_{n\to\infty} \mathcal{T}^n = U \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} V^{+}$$

$$= \vec{\pi}\,\vec{1}^{+}$$

$$e^{\mathcal{R} n \Delta t} \to \pi\,\vec{1}^{+}$$

same as the diagonalizable case

In the diagonalizable case, we can also understand relaxation to $\vec{\pi}$ as follows:

$$\vec{p}(0) = \sum_{n=0}^{N-1} c_n \vec{u}_n = c_0 \vec{\pi} + \sum_{n=1}^{N-1} c_n \vec{u}_n$$

Normalize:

$\vec{v}_0^{\,\dagger}$

$$1 = \vec{1}^{\dagger} \cdot \vec{p}(0) = c_0 \underbrace{\vec{1}^{\dagger} \cdot \vec{\pi}}_{1} + \sum_{n>0} c_n \underbrace{\vec{1}^{\dagger} \cdot \vec{u}_n}_{0}$$

$$\therefore \boxed{c_0 = 1}$$

$$\vec{p}(0) = \vec{\pi} + \sum_{n>0} c_n \vec{u}_n$$

$\lambda_0 = 0$, $\mathrm{Re}\,\lambda_n < 0$

$$\vec{p}(t) = e^{\mathcal{R}t} \vec{p}(0) = e^{\lambda_0 t} \vec{\pi} + \sum_{n>0} c_n e^{\lambda_n t} \vec{u}_n \xrightarrow{t \to \infty} \vec{\pi}$$

For the non-diagonalizable case, see problem #3 of Problem Set 4.